

# Near field imaging of small isotropic and extended anisotropic scatterers

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## Abstract

In this paper, we consider two time-harmonic inverse scattering problems of reconstructing penetrable inhomogeneous obstacles from near field measurements. First we appeal to the Born approximation for reconstructing small isotropic scatterers via the MUSIC algorithm. Some numerical reconstructions using the MUSIC algorithm are provided for reconstructing the scatterer and piecewise constant refractive index using a Bayesian method. We then consider the reconstruction of an anisotropic extended scatterer by applying the factorization method to the near field operator. This provides a rigorous characterization of the support of the anisotropic obstacle in terms of a range test derived from the measured data. Under appropriate assumptions on the material parameters, the derived factorization can be used to determine the support of the medium without a-priori knowledge of the material properties.

**Keywords:** inverse scattering, factorization method, MUSIC algorithm.

**AMS subject classifications:** 35J05, 45Q05, 78A46

## 1 Introduction

Qualitative/Sampling methods have been used to solve multiple inverse problems such as parameter identification and shape reconstruction. These methods have been used to solve inverse boundary value problems for elliptic, parabolic and hyperbolic partial differential equation. See [15] and [14] for examples of the linear sampling methods applied to parabolic and hyperbolic equations, respectively. The vast available literature is a representative of the many directions that this research has taken (see [2], [9], [10], [18] and the references therein).

The factorization method is a qualitative method which develops a range test to determine the support of the unknown scatterer. In general, we have that

$$\phi_z \in \text{Range}(N_{\sharp}^{1/2}) \iff z \text{ is inside the obstacle}$$

where  $\phi_z$  is known and  $N_{\sharp}$  is a positive selfadjoint compact operator defined by the measurements. By appealing to the range test, we have that any two obstacles admitting the same data must be equal, proving uniqueness of the inverse problem. Since the support

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of the scatterer is connected to the range of a compact operator which is known, evaluating the indicator function derived from the factorization method amounts to applying Picards criteria, which only requires the singular values and functions of a known operator. This makes the factorization method computationally cheap to implement and analytically rigorous.

In this paper, we consider two problems associated with inverse scattering. The first problem we consider is the reconstruction of small isotropic obstacles and the refractive index from near field data. To do so, we derive the Multiple-Signal-Classification (MUSIC) algorithm which gives an analytically rigorous method of reconstructing small obstacles. Once we have reconstructed the obstacles, we will give a few example of how one can reconstruct constant refractive indices using a Bayesian method. Next we consider the reconstructing an anisotropic obstacle from near field data via the factorization method. This problem has been considered in [3] and [19] for far field measurements using the linear sampling and factorization methods, respectively. The factorization method for an inhomogeneous isotropic media with near field measurements was studied in [16]. We analyze the factorization method applied to the near field operator for an anisotropic obstacle to derive a simple numerical algorithm to reconstruct the scatterer. For the case of an anisotropic media, the unique determination of the support is optimal because a matrix-valued coefficient is in general not uniquely determined by the scattering data (see for e.g. [12]). The analysis done in this paper is of a similar flavor to the work done in [6] and [22].

## 2 Recovering small isotropic scatterers and the refractive index

### 2.1 Scattering by small isotropic scatterers

In this section, we formulate the direct time-harmonic scattering problem in  $\mathbb{R}^d$  for small isotropic scatterers. The scattering obstacle may be made up of multiple simply components, but is not necessarily simply connected. We are interested in using near field measurements for ‘small’ obstacles embedded in a homogenous media. Let  $u^i(\cdot, y)$  be the incident field that originates from the source point  $y$  located on the curves  $C$ . We will assume that the curve  $C$  is class  $\mathcal{C}^2$  and that the incident field satisfies the Helmholtz equation in  $\mathbb{R}^d \setminus \{y\}$  for  $d = 2$  or  $3$ . Now let

$$D = \bigcup_{m=1}^M D_m \quad \text{with} \quad D_m = (z_m + \varepsilon B_m),$$

where  $\varepsilon > 0$  is a small parameter and  $B_m$  is a domain with a piece-wise smooth boundary that is centered at the origin. Outside of the scattering obstacle(s)  $D_m$ , the material

parameters are homogeneous isotropic with refractive index scaled to one. Here we denote  $n(x)$  the refractive index of the homogenous background with the (possibly inhomogeneous) obstacle  $D$  in  $\mathbb{R}^d$  by

$$n(x) = 1(1 - \chi_D) + \sum_{m=1}^M n_m(x) \chi_{D_m}$$

where  $\chi$  is the characteristic function. The refractive index of the obstacle(s)  $D_m$  is represented by a continuous scalar function  $n_m$  such that the real and imaginary parts satisfy

$$\inf_{m=1 \dots M} \Re(n_m(x)) \geq n_{min} > 0 \quad \text{and} \quad \Im(n_m) \geq 0$$

for almost every  $x \in D_m$  where we assume that  $n_{min} > 1$  or  $n_{min} < 1$ . Now the radiating scattered field  $u^s(\cdot, y) \in H_{loc}^1(\mathbb{R}^d)$  given by the point source incident field  $u^i(\cdot, y) = \Phi(\cdot, y)$  is the unique solution to

$$\Delta_x u^s + k^2 n u^s = k^2 (1 - n) u^i \quad \text{in} \quad \mathbb{R}^d \quad (1)$$

$$\partial_r u^s - i k u^s = \mathcal{O}\left(\frac{1}{r^{(d+1)/2}}\right) \quad \text{as} \quad r \rightarrow \infty \quad (2)$$

where  $r = |x|$  and the radiation condition (2) is satisfied uniformly in all directions  $\hat{x} = x/|x|$ . Recall the fundamental solution to Helmholtz equation

$$\Phi(x, y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k|x-y|) & \text{in } \mathbb{R}^2 \\ \frac{1}{4\pi} \frac{\exp(ik|x-y|)}{|x-y|} & \text{in } \mathbb{R}^3, \end{cases}$$

where  $H_0^{(1)}$  is the first kind Hankel function of order zero. The corresponding total field for the scattering problem is given by  $u(\cdot, y) = u^s(\cdot, y) + u^i(\cdot, y)$  satisfies

$$u^{(-)} = u^{(+)} \quad \text{and} \quad \frac{\partial u^{(-)}}{\partial \nu} = \frac{\partial u^{(+)}}{\partial \nu} \quad \text{on } \partial D_m,$$

where the superscripts  $+$  and  $-$  for a generic function indicates the trace on the boundary taken from the exterior or interior of its surrounding domain, respectively. We have that the scattered field is given by the solution to the Lippmann-Schwinger Integral Equation (see for e.g. [9])

$$u^s(x, y) = k^2 \int_D (n(z) - 1) \Phi(x, z) u(z, y) dz.$$

Assume that  $\varepsilon$  is sufficiently small such that

$$k^2 \left| \int_D (n(z) - 1) \Phi(x, z) dz \right| \ll 1.$$

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Therefore, we can conclude that the Born approximation, which is the first term in the Neumann series solution to the Lippmann-Schwinger Integral Equation, given by

$$u_B^s(x, y) = k^2 \int_D (n(z) - 1) \Phi(x, z) u^i(z, y) dz \quad (3)$$

is a ‘good’ approximation for the scattered field  $u^s(x, y)$ . Therefore, we assume that we have the measured near field data that is given by the Born approximation  $u_B^s(x, y)$ . Let  $\Omega$  be the bounded simply connected open set such that satisfies  $\overline{D} \subset \Omega$  and  $C = \partial\Omega$ . Now assume that we have the data set of near field measurements  $u_B^s(x, y)$  for all  $x, y \in C$ . The *inverse problem* we consider is to reconstruct the scatterer  $D$  and the refractive index  $n$  from a knowledge of the Born approximation of the near field measurements.

## 2.2 The MUSIC Algorithm

The MUSIC Algorithm can be considered as the discrete analogue of the factorization method (see for e.g. [8]). In particular, we connect the locations of the obstacles  $D_m$  given by  $\{z_m : m = 1, \dots, M\}$  to the range of the so-called multi-static response matrix denoted by  $\mathbf{N}$  that is defined by the near field measurements  $u_B^s(x, y)$  for all  $x, y \in C$ . To arrive at the MUSIC algorithm, we will need to define our measurements operator. To this end, we let the incident field be give by  $u^i(x, y) = \Phi(x, y)$ . Notice that by (3) we obtain that

$$\begin{aligned} u_B^s(x, y) &= k^2 \int_D (n(z) - 1) \Phi(x, z) \Phi(z, y) dz \\ &= \sum_{m=1}^M k^2 (n(z_m) - 1) \varepsilon^d |B_m| \Phi(x, z_m) \Phi(z_m, y) + o(\varepsilon^d) \end{aligned}$$

since  $x, y \in C$  the integrand is continuous with respect to the variable  $z \in D$ . Assume we have a finite number of incident and observation directions where  $N \geq M$ , with  $M$  being the number of small obstacles and  $N$  being the number of incident and observation directions given by  $x_i, y_j \in C$ . Now define the multi-static response matrix

$$\mathbf{N}_{i,j} = \sum_{m=1}^M k^2 (n(z_m) - 1) \varepsilon^d |B_m| \Phi(x_i, z_m) \Phi(z_m, y_j).$$

Notice that for  $\varepsilon$  sufficiently small we have that the multi-static response matrix is approximated by  $u_B^s(x_i, y_j)$  therefore we assume that  $\mathbf{N}$  is known. For convenience, we will assume that the sources and receivers are placed at the same points (i.e.  $x_i = y_i$ ). We now factorize the multi-static response matrix, therefore we define the matrices  $\mathbf{U} \in \mathbb{C}^{N \times M}$  and  $\mathbf{\Sigma} \in \mathbb{C}^{M \times M}$ , where the matrix  $\mathbf{U}$  is given by

$$\mathbf{U}_{i,m} = \Phi(x_i, z_m)$$

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and the matrix  $\mathbf{\Sigma} = \text{diag}(\sigma_m)$  where  $\sigma_m = k^2(n(z_m) - 1)\varepsilon^d|B_m| \neq 0$ . Therefore, we have that  $\mathbf{N} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^\top$  and it follows that

$$\mathbf{N}\mathbf{N}^* = \mathbf{U}\tilde{\mathbf{\Sigma}}\mathbf{U}^* \quad \text{with} \quad \tilde{\mathbf{\Sigma}} = \mathbf{\Sigma}\mathbf{U}^\top \overline{\mathbf{U}}\mathbf{\Sigma}^*. \quad (4)$$

Now define the vector  $\phi_z \in \mathbb{C}^N$  for any point  $z \in \mathbb{R}^d$  by

$$\phi_z = \left( \Phi(x_1, z), \dots, \Phi(x_N, z) \right)^\top. \quad (5)$$

The goal of this section is to show that  $\phi_z$  is in the range of the matrix  $\mathbf{N}\mathbf{N}^*$  if and only if  $z \in \{z_m : m = 1, \dots, M\}$ , which is a discrete reformulation of the result of the factorization method stated in the Introduction. Since we are interested in finding obstacles  $D_m$ , it is sufficient to prove the result only for values of  $z \in \Omega$ . Recall that without loss of generality, we assume that the sources and receivers are placed at the same locations. We will follow the analytic framework in Section 4.1 of [18]. We now give a result that can be proven by using standard arguments from linear algebra (see proof of Theorem 3.1 in [11] for details).

**Lemma 2.1.** *Let the matrix  $\mathbf{N}$  have the following factorization  $\mathbf{N} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^\top$  where  $\mathbf{U} \in \mathbb{C}^{N \times M}$  and  $\mathbf{\Sigma} \in \mathbb{C}^{M \times M}$  with  $N > M$ . Assume that*

1. *the matrix  $\mathbf{U}$  has full rank  $M$*
2. *the matrix  $\mathbf{\Sigma}$  is invertible*

*then  $\text{Range}(\mathbf{U}) = \text{Range}(\mathbf{N}\mathbf{N}^*)$ .*

We now construct an indicator function derived from Lemma 2.1 to reconstruct the locations of the scattering obstacles. To this end, for each sampling point  $z \in \Omega$  we will show that the vector  $\phi_z$  is in the range of  $\mathbf{N}\mathbf{N}^*$  if and only if  $z \in \{z_m : m = 1, \dots, M\}$ . We now prove an auxiliary result that connects the location of the scattering obstacles to the range of the matrix  $\mathbf{U}$ .

**Theorem 2.1.** *Assume that the set  $S = \{x_i \in C : i \in \mathbb{N}\}$  is dense in  $C$ . Let  $z \in \Omega$  then there is a number  $N_0 \in \mathbb{N}$  such that for all  $N \geq N_0$ , we have that*

1. *the matrix  $\mathbf{U}$  has full rank,*
2.  *$\phi_z \in \text{Range}(\mathbf{U})$  if and only if  $z \in \{z_m : m = 1, \dots, M\}$ .*

*Proof.* It is clear that  $\phi_{z_m}$  is in the range of  $\mathbf{U}$  since  $\phi_{z_m}$  is the  $m$ -th column of  $\mathbf{U}$ . To prove that for  $z \in \Omega \setminus \{z_m : m = 1, \dots, M\}$  that  $\phi_z$  is not in the range of  $\mathbf{U}$  for some  $N$  sufficiently large, we proceed by way of contradiction. Let  $z \in \Omega \setminus \{z_m : m = 1, \dots, M\}$

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and assume that there does not exist such a  $N_0$ , then we have that there exists  $\alpha_m^N, \alpha^N \in \mathbb{C}$  such that

$$|\alpha^N| + \sum_{m=1}^M |\alpha_m^N| = 1 \quad (6)$$

and

$$\alpha^N \Phi(x_i, z) + \sum_{m=1}^M \alpha_m^N \Phi(x_i, z_m) = 0 \quad \text{for all } 1 \leq i \leq N.$$

We then conclude that (up to a subsequence)  $\alpha_m^N, \alpha^N \rightarrow \alpha_m, \alpha$  as  $N \rightarrow \infty$ . This gives that due to the density of  $S$  and since  $\Phi(x, z)$  is analytic with respect to  $x$  for  $z \neq x$ , we obtain that

$$\alpha \Phi(x, z) + \sum_{m=1}^M \alpha_m \Phi(x, z_m) = 0 \quad \text{for all } x \in C.$$

Since the mapping

$$x \mapsto \alpha \Phi(x, z) + \sum_{m=1}^M \alpha_m \Phi(x, z_m)$$

is a radiation exterior solution to the Helmholtz equation in  $\mathbb{R}^d \setminus \overline{\Omega}$  we conclude that

$$\alpha \Phi(x, z) + \sum_{m=1}^M \alpha_m \Phi(x, z_m) = 0 \quad \text{for all } x \in \mathbb{R}^d \setminus \{z\} \cup \{z_m : m = 1, \dots, M\}$$

by uniqueness to the exterior Dirichlet problem and unique continuation. Now letting  $x \rightarrow z, z_m$  gives that  $\alpha_m = \alpha = 0$  for  $1 \leq m \leq M$ , but by (6) and the convergence of the coefficients  $\alpha_m^N, \alpha^N$  as  $N \rightarrow \infty$  we have that

$$|\alpha| + \sum_{m=1}^M |\alpha_m| = 1$$

which gives a contradiction. Moreover, the fact that  $\mathbf{U}$  has full rank is a consequence of the given arguments.  $\square$

Now by combining this with Lemma 2.1 and Theorem 2.1 we have the following range test which can be used to reconstruct the set  $z \in \{z_m : m = 1, \dots, M\}$ .

**Theorem 2.2.** *Assume that the set  $S = \{x_i \in C : i \in \mathbb{N}\}$  is dense in  $C$ . If  $z \in \Omega$  then there is a number  $N_0 \in \mathbb{N}$  such that for all  $N \geq N_0$*

$$\phi_z \in \text{Range}(\mathbf{N}\mathbf{N}^*) \quad \text{if and only if} \quad z \in \{z_m : m = 1, \dots, M\}.$$

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Now we are ready to construct an indicator function which characterizes the locations of the small obstacles. To this end, let  $\mathbf{P} : \mathbb{C}^N \mapsto \text{Null}(\mathbf{N}\mathbf{N}^*)$  be the orthogonal projection onto  $\text{Null}(\mathbf{N}\mathbf{N}^*)$ . Therefore, by Theorem 2.2 we have that

$$\mathbf{P}\phi_z = 0 \quad \text{if and only if} \quad z \in \{z_m : m = 1, \dots, M\}.$$

Notice that since  $\mathbf{N}\mathbf{N}^*$  is a self-adjoint matrix we have that it is orthogonally diagonalizable. So we let  $\mathbf{w}_j$  be the  $j$ -th orthonormal eigenvector of  $\mathbf{N}\mathbf{N}^*$ , we also let  $r = \text{Rank}(\mathbf{N}\mathbf{N}^*)$ . This gives that  $\mathbf{w}_j$  for  $r+1 \leq j \leq N$  is an orthonormal basis for  $\text{Null}(\mathbf{N}\mathbf{N}^*)$ , and therefore we have that  $\mathbf{P}$  is given by

$$\mathbf{P}\phi_z = \sum_{j=r+1}^N (\phi_z, \mathbf{w}_j)_{\ell^2} \mathbf{w}_j.$$

This leads to the following result:

**Lemma 2.2.** *Let  $\mathbf{w}_j$  be the  $j$ -th orthonormal eigenvector of  $\mathbf{N}\mathbf{N}^*$  and let  $r = \text{Rank}(\mathbf{N}\mathbf{N}^*)$ . Assume that the set  $S = \{x_i \in C : i \in \mathbb{N}\}$  is dense in  $C$ . If  $z \in \Omega$  then there is a number  $N_0 \in \mathbb{N}$  such that for all  $N \geq N_0$*

$$\mathcal{I}(z) = \left[ \sum_{j=r+1}^N |(\phi_z, \mathbf{w}_j)|_{\ell^2}^2 \right]^{-1} < \infty \quad \text{if and only if} \quad z \in \{z_m : m = 1, \dots, M\}. \quad (7)$$

### 2.3 Numerical validation of the MUSIC algorithm

We now present a few examples of reconstructing small obstacles using the MUSIC algorithm developed in this section in  $\mathbb{R}^2$ . To simulate the data, we use the Born approximation of the scattered field and a  $2d$  Gaussian quadrature to approximate the volume integral. We now let the collection curve  $C$  be the boundary of a disk of radius one. In the following calculations we use 32 different sources and receivers at the points

$$x_i = y_i = (\cos(2\pi(i-1)/32), \sin(2\pi(i-1)/32))^\top \quad \text{for} \quad i = 1, \dots, 32.$$

This leads to an approximated multi-static response matrix

$$\mathbf{N} \approx [u_B^s(x_i, y_j)]_{i,j=1}^{32} \quad \text{such that} \quad u_B^s(x, y) = k^2 \sum_{\ell} \omega_{\ell} (n(z_{\ell}) - 1) \Phi(x, z_{\ell}) \Phi(z_{\ell}, y),$$

where  $\omega_{\ell}$  and  $z_{\ell}$  are the weights and quadrature points for the numerical approximation of the volume integral by a  $2d$  Gaussian quadrature method. We give examples with random noise added to the simulated data for  $u_B^s(x_i, y_j)$ . The random noise level is given by  $\delta$  where the noise is added to the multi-static response matrix such that

$$[u_B^s(x_i, y_j)(1 + \delta E_{i,j})]_{i,j=1}^{32} \quad \text{with the random matrix} \quad E \quad \text{satisfying} \quad \|E\|_2 = 1.$$

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The set  $\{z_m : m = 1, \dots, M\}$  can be visualized by plotting the corresponding indicator function given in (7)

$$\mathcal{I}(z) = \left[ \sum_{j=r+1}^{32} |(\phi_z, \mathbf{w}_j)|_{\ell^2}^2 \right]^{-1}$$

where  $\mathbf{w}_j \in \mathbb{C}^{32}$  are the eigenvectors of the approximated matrix  $\mathbf{N}\mathbf{N}^*$  and  $r = \text{Rank}(\mathbf{N}\mathbf{N}^*)$  is computed by the Matlab command rank. In the following examples, we fix the wave number  $k = 1$ .

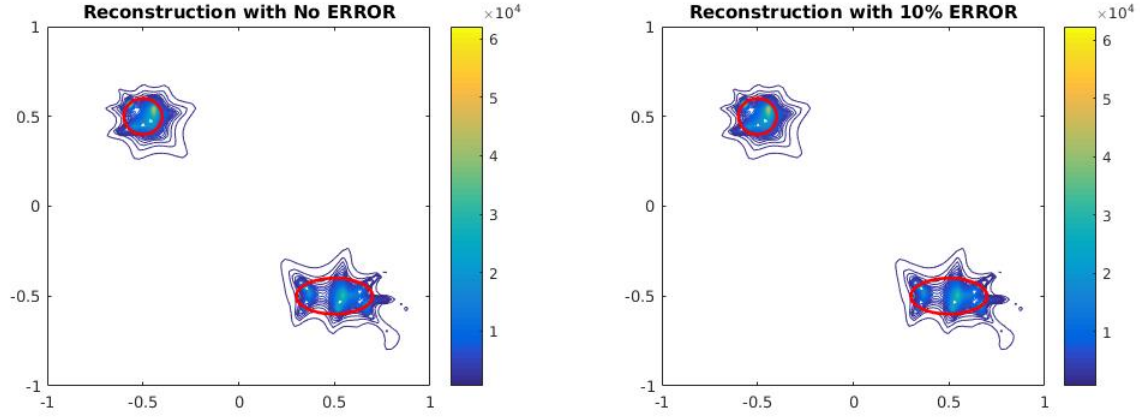


Figure 1: Reconstruction of the location of the two scatterers, the circle of radius 0.2 centered at  $(-0.5, 0.5)$  and the ellipse with  $a = 0.2$  and  $b = 0.1$  centered at  $(0.5, -0.5)$ . The refractive index  $n = 5$  in both scatterers. Contour plot of the indicator function  $z \mapsto \mathcal{I}(z)$  where the red lines are the boundary of the scatterer.



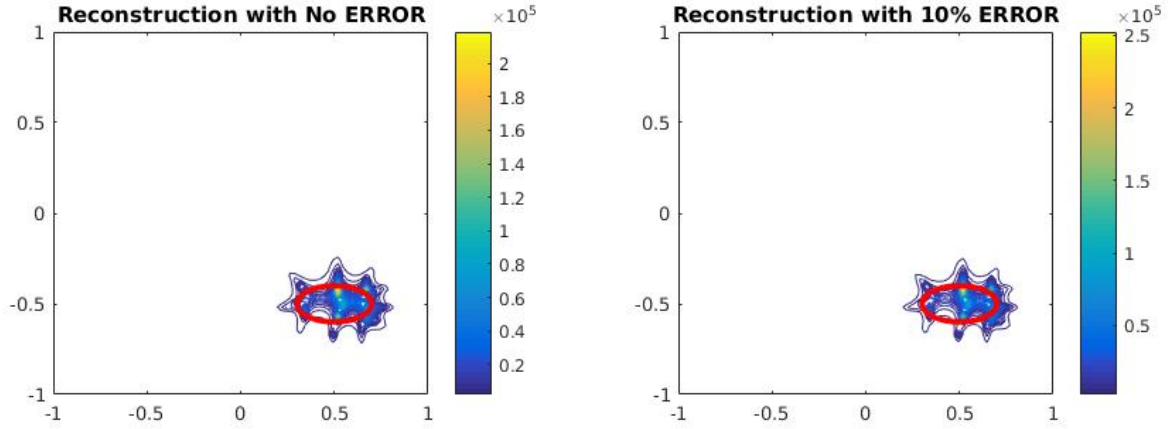


Figure 2: Reconstruction of the location of one scatterer, the ellipse with  $a = 0.2$  and  $b = 0.1$  centered at  $(0.5, -0.5)$ . The refractive index  $n = 2 + i$  in the scatterer. Contour plot of the indicator function  $z \mapsto \mathcal{I}(z)$  where the red lines are the boundary of the scatterer.

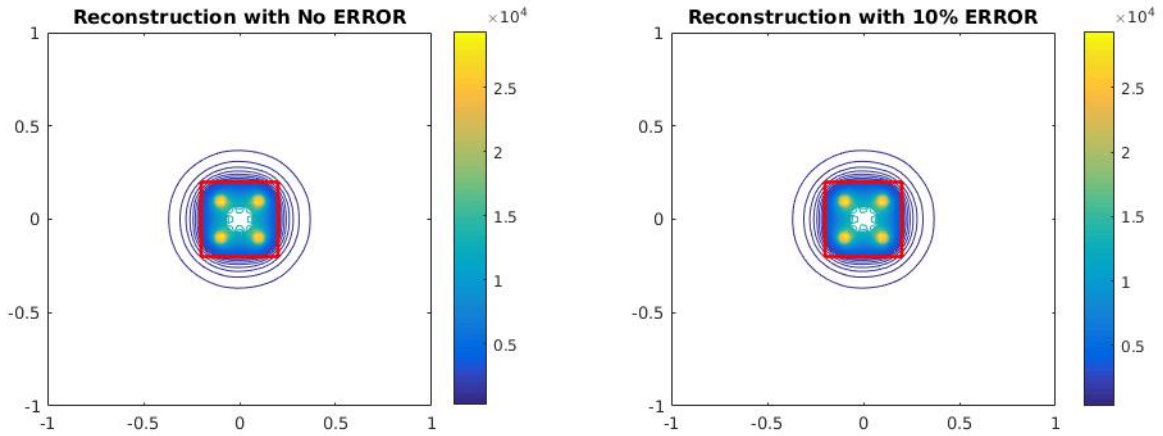


Figure 3: Reconstruction of the location of the a square scatterer,  $[-0.2, 0.2]^2$ . The refractive index  $n(x_1, x_2) = x_1^2 + 2$  in the scatterer. Contour plot of the indicator function  $z \mapsto \mathcal{I}(z)$  where the red lines are the boundary of the scatterer. *Remark:* Even though both figures look identical, different data we used in the reconstructions.

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## 2.4 Recovery of the refractive index

In this section, we consider the *inverse problem* of reconstructing the piece-wise constant refractive index from the measured scattered field. To this end, we provide some numerical examples of reconstructing the refractive index using a Bayesian statistical approach for small isotropic scatterers. Here we assume that the scatterer and the (Born approximation of the) scattered field is known. From the previous section, we see that the support of the scatterer can be reconstructed by the MUSIC algorithm. One can also use the factorization method or linear sampling methods to reconstruct the support of  $D$ .

We will now assume that  $n$  is piecewise constant. Rather than solving for the value of the unknown  $n$  exactly, we solve for a probability distribution of the possible values of  $n$ . To do this, we must assume a prior distribution for possible values of  $n$  and a data model for the scattered wave, which we will discuss below. Then, Markov chain Monte Carlo methods allow us to sample from the posterior distribution of  $n$  given the scattered wave and the prior distribution. The mode of the resulting posterior distribution (called the Maximum a posteriori estimator or MAP) can be shown to be equivalent to the standard linear regression estimator in special cases; however, in this context, our approach provides more information on possible values of  $n$  than linear regression.

Similar approaches utilizing Bayesian statistics and the Born approximation have been used when applying compressive sensing techniques to inverse scattering [26],[27],[28],[29]. In those context, there is an a-priori assumption on the sparsity of the scatterer. The approach has also been applied to continuous random media [30]. However, in this case, the scatterer is not sparse and it is not a continuous random medium. Moreover, we extend the approach by using the MUSIC algorithm to find the support of the scatterer, and use that information in our recovery of the refractive index.

In our experiments, we assume that the measured Born approximation of the scattered field denoted  $U_B^s$  is given by

$$U_B^s(x, y) = k^2 \int_D (n(z) - 1) \Phi(x, z) \Phi(z, y) dz + \mathcal{O}(h).$$

where  $h$  is the amount of error added for the reconstruction of  $D$  and standard measurement error. We assume the error term is modeled by a normal distribution with zero mean and standard deviation  $h$ , then we may write the data model, as a statistical distribution where values of  $U_B^s$  are drawn from

$$U_B^s(x, y) \sim k^2 \int_D (n(z) - 1) \Phi(x, z) \Phi(z, y) dz + \mathcal{N}(0, h). \quad (8)$$

This gives that the measured  $U_B^s$  is normally distributed with mean given by the actual value of the Born approximation  $u_B^s$  to the scattered field and standard deviation given by the error level  $h$ .

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Now, assume that the scatterer  $D = D_1 \cup D_2$  where  $D_1 \cap D_2$  is empty. Further, we assume  $n$  is constant on  $D_1$  and  $D_2$ , e.g.  $n = n_1$  on  $D_1$  and  $n = n_2$  on  $D_2$  for  $n_i \in \mathbb{C}$ . In the following experiments we let  $D_1$  be the circle of radius 0.2 centered at  $(-0.5, 0.5)$  and  $D_2$  be the ellipse with  $a = 0.2$  and  $b = 0.1$  centered at  $(0.5, -0.5)$ . A typical technique in Bayesian statistics is to assume a prior distribution for unknown values, in this case,  $n_1$  and  $n_2$ . The full Bayesian model for  $U_B^s$  at the measurement points  $x_i, y_j \in C$  is given by the equations

$$U_B^s \sim \mathcal{N}(\mu, h)$$

where

$$\mu = k^2 \int_D (n(z) - 1) \Phi(x, z) \Phi(z, y) dz$$

with prior distributions

$$\text{Im}(n_1) \sim \mathcal{N}(0, \gamma), \quad \text{Im}(n_2) \sim \mathcal{N}(0, \gamma),$$

$$\text{Re}(n_1) \sim \mathcal{N}(0, \gamma), \quad \text{Re}(n_2) \sim \mathcal{N}(0, \gamma), \quad \text{and} \quad h \sim \mathcal{U}(0, H),$$

where  $\gamma$  and  $H$  are on the order of 100. We choose  $\gamma$  and  $H$  this way for practical reasons. Traditionally, with no previous data one selects a flat prior distribution, but numerically modeling such an assumption as a Normal distribution with a large standard deviation is convenient. It is important to draw the distinction that the distribution for the real and imaginary parts of  $n_1$  and  $n_2$  are separate. There is no ‘‘partial pooling’’ in our formulation as we do not assume the components of each  $n_i$  are drawn from the same distribution. For our purposes, we may simplify the above process by only using the real or imaginary part of  $U_B^s$  and  $\mu$ .

In Bayesian notation, we have that the posterior distribution for  $n_1$ ,  $n_2$ , and  $h$  are distributed according to the proportion:

$$P(n_1, n_2, h | U_B^s) \propto \ell(U_B^s | n_1, n_2, h) P(n_1) P(n_2) P(h), \quad (9)$$

where  $\ell$  is the likelihood function arising from the data model for  $U_B^s$  and  $P(n_i)$ ,  $P(h)$  are the assumed priors given above. One may marginalize over  $h$  to obtain  $P(n_1, n_2 | U_B^s)$  if desired. By using the Metropolis-Hastings algorithm, we can derive this posterior distribution, and thus values for  $n_1$  and  $n_2$ . For our calculation, we use the implementation in the Python package PyMC3 [25].

For each experiment, we plot the posterior probability distribution for values of  $n_1$  and  $n_2$  with wavenumber  $k = 1$ . We perform an experiment without noise and with 10% noise for different  $n_1, n_2$  pairs. Note, we used standard techniques when dealing with sampling data for the plots, including taking every  $p$ -th sample to ensure the resulting sample is not intercorrelated. Each experiment consists of 30000 total samples, which includes a burn in period of 10000 samples. We arbitrarily chose  $p = 20$ .

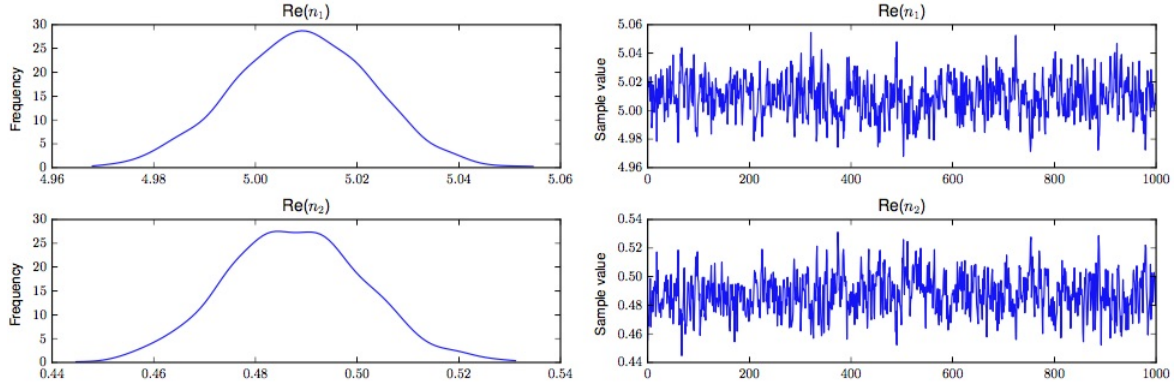


Figure 4: Recovery with 10% Gaussian noise applied to the measurements in the case when  $n_1 = 5$  and  $n_2 = 0.5$ . The left hand images shows the real part of posterior distribution for  $n_1$  and  $n_2$ . The right hand plots show the sample values (y-axis) for the  $i$ -th sample (x-axis).

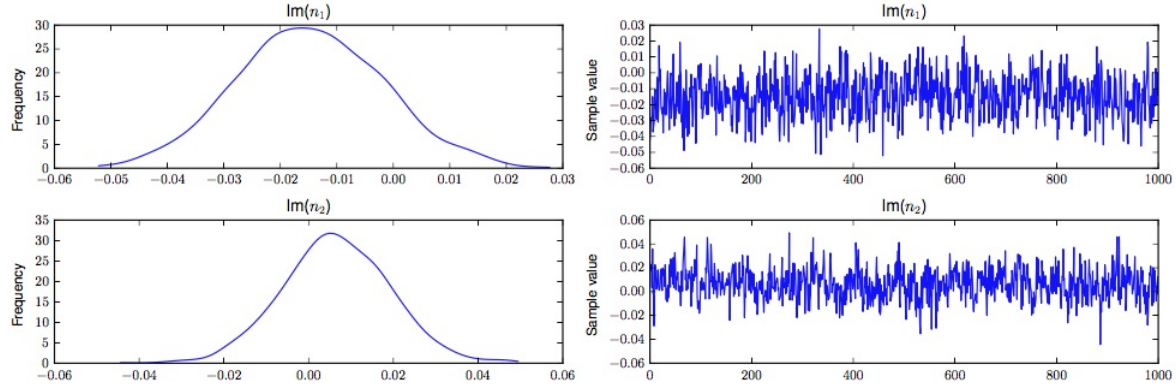


Figure 5: Recovery with 10% Gaussian noise applied to the measurements in the case when  $n_1 = 5$  and  $n_2 = 0.5$ . The left hand images shows the imaginary part of posterior distribution for  $n_1$  and  $n_2$ . The right hand plots show the sample values (y-axis) for the  $i$ -th sample (x-axis).

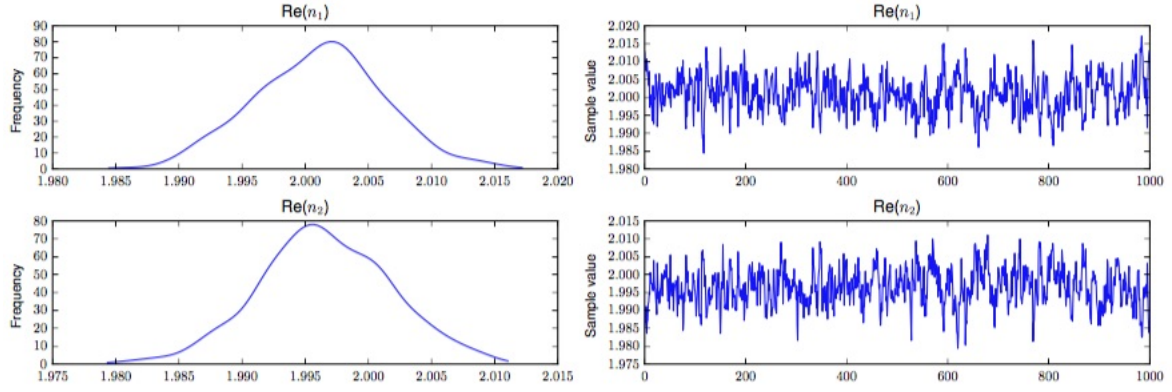


Figure 6: Recovery with 10% Gaussian noise applied to the measurements in the case when  $n_1 = n_2 = 2 + i$ . The left hand images shows the real part of posterior distribution for  $n_1$  and  $n_2$ . The right hand plots show the sample values (y-axis) for the  $i$ -th sample (x-axis).

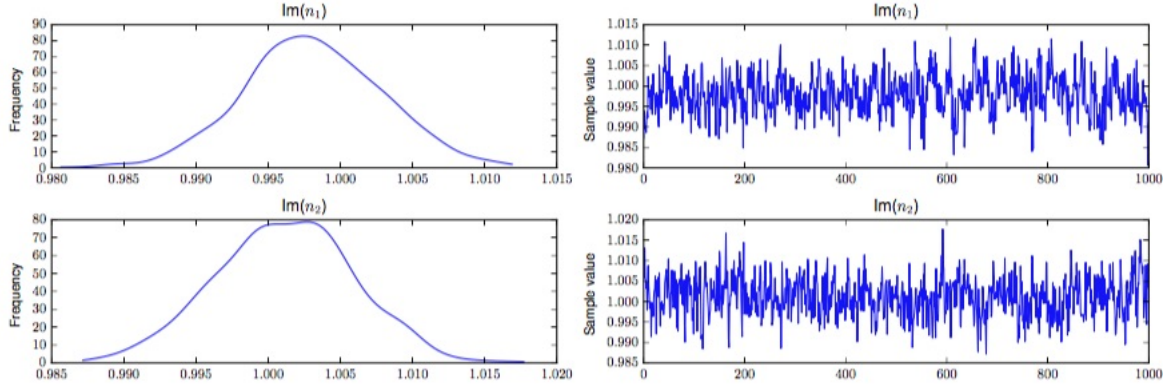


Figure 7: Recovery with 10% Gaussian noise applied to the measurements in the case when  $n_1 = n_2 = 2 + i$ . The left hand images shows the imaginary part of posterior distribution for  $n_1$  and  $n_2$ . The right hand plots show the sample values (y-axis) for the  $i$ -th sample (x-axis).

### 3 The Factorization Method for anisotropic scatterers

#### 3.1 Scattering by extended anisotropic scatterers

In this section we formulate the direct and inverse time-harmonic scattering problems under consideration. To this end, we let the point source located at the point  $y \in C$  be given by the fundamental solution to Helmholtz equation  $\Phi(x, y)$  where  $C$  is assumed to be a closed curve (in  $\mathbb{R}^2$ ) or surface (in  $\mathbb{R}^3$ ) that is class  $\mathcal{C}^2$  or sufficiently smooth, such that  $H^{1/2}(C)$  is compactly embedded in  $L^2(C)$ . We consider the scattering of the (non-standard) incident field  $u^i(\cdot, y) = \overline{\Phi(\cdot, y)}$  for  $y \in C$  by a penetrable anisotropic obstacle. Assume that the obstacle  $D \subset \mathbb{R}^d$  (for  $d = 2, 3$ ) is a bounded simply connected open set having piece-wise smooth boundary  $\partial D$  with  $\nu$  being the unit outward normal to the boundary.

We assume that the constitutive parameters of the scatterer  $D$  are represented by a symmetric matrix  $\tilde{A} \in \mathcal{C}^2(D, \mathbb{C}^{d \times d})$  and a scalar function  $\tilde{n} \in \mathcal{C}(D)$  such that

$$\bar{\xi} \cdot \Re(\tilde{A}(x)) \xi \geq a_{\min} |\xi|^2 > 0 \quad \text{and} \quad \bar{\xi} \cdot \Im(\tilde{A}(x)) \xi \leq 0,$$

where as

$$\Re(\tilde{n}(x)) \geq n_{\min} > 0 \quad \text{and} \quad \Im(\tilde{n}(x)) \geq 0$$

for almost every  $x \in D$  and all  $\xi \in \mathbb{C}^d$ . Outside the obstacle  $D$ , the material parameters are homogeneous isotropic with refractive index scaled to one. We denote by  $A$  and  $n$  the constitutive parameters of the homogenous background with the anisotropic obstacle  $D$  in  $\mathbb{R}^d$  by

$$A(x) = \tilde{A}(x)\chi_D + I(1 - \chi_D) \quad \text{and} \quad n(x) = \tilde{n}(x)\chi_D + 1(1 - \chi_D)$$

where  $I$  denotes the  $d \times d$  identity matrix and  $\chi_D$  is the characteristic function on  $D$ . Note that the support of the contrasts  $A - I$  and  $n - 1$  is  $\overline{D}$ . The radiating scattered field  $u^s(\cdot, y)$  given by the (non-standard) point source  $u^i(\cdot, y)$  satisfies

$$\nabla_x \cdot A \nabla_x u^s + k^2 n u^s = \nabla_x \cdot (I - A) \nabla_x u^i + k^2 (1 - n) u^i \quad \text{in } \mathbb{R}^d \quad (10)$$

$$\partial_r u^s - i k u^s = \mathcal{O}\left(\frac{1}{r^{(d+1)/2}}\right) \quad \text{as } r \rightarrow \infty, \quad (11)$$

where  $r = |x|$  and (11) is satisfied uniformly in all directions  $\hat{x} = x/|x|$ . It can be shown that the scattering problem (10)-(11) has a unique solution  $u^s \in H_{loc}^1(\mathbb{R}^d)$  by using a variational approach (see for e.g. [2]). The total field corresponding to the scattering problem (10) is defined as  $u(\cdot, y) = u^s(\cdot, y) + u^i(\cdot, y)$  and satisfies

$$u^{(-)} = u^{(+)} \quad \text{and} \quad \frac{\partial u^{(-)}}{\partial \nu_A} = \frac{\partial u^{(+)}}{\partial \nu} \quad \text{on } \partial D,$$

where the superscripts  $+$  and  $-$  for a generic function indicates the trace on the boundary taken from the exterior or interior of the domain, respectively. Now assume that we have

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the data set of near field measurements  $u^s(x, y)$  for all  $x, y \in C$  and define the Near Field operator

$$N : L^2(C) \mapsto L^2(C) \quad \text{given by} \quad (Ng)(x) = \int_C u^s(x, y)g(y) ds_y.$$

The *inverse problem* we consider is to reconstruct the scattering obstacle  $D$  from a knowledge of  $u^s(x, y)$  for all  $x, y \in C$ . We will let  $\Omega$  be the bounded simply connected open set such that  $\overline{D} \subset \Omega$  and  $C = \partial\Omega$ .

In the coming sections, we will see that to obtain a symmetric factorization of the near field operator we require that the incident wave be the complex conjugate of a point source. However, it is known that these non-physical sources can be approximated by a linear combination of physical point sources (see [17]). In [16], the authors are able to avoid using non-physical sources but must construct the so-called outgoing-to-incoming operator. If  $C$  is given by the boundary of a disk/sphere centered at the origin, then the outgoing-to-incoming operator is given by a boundary integral operator that does not depend on the type of scatterer.

### 3.2 Factorization of the near field operator

This section is dedicated to constructing a suitable factorization of the near field operator  $N$  so that we are able to use the theoretical results in [18] and/or [20], in order to derive an indicator function for the support of  $D$  in terms of the near field measurements. To this end, motivated by the source problem given in equation (10)-(11) for the scattered field due to the (non-standard) point source, we consider the problem of finding  $u \in H_{loc}^1(\mathbb{R}^d)$  for a given  $v \in H^1(D)$  such that

$$\begin{aligned} \nabla \cdot A \nabla u + k^2 n u &= \nabla \cdot (I - A) \nabla v + k^2 (1 - n) v \quad \text{in } \mathbb{R}^d \\ \partial_r u^s - i k u^s &= \mathcal{O}\left(\frac{1}{r^{(d+1)/2}}\right) \quad \text{as } r \rightarrow \infty. \end{aligned} \tag{12}$$

It can be shown that for any given  $v$ , the solution operator that maps  $v \mapsto u$  is a bounded linear mapping from  $H^1(D)$  to  $H_{loc}^1(\mathbb{R}^d)$ . At this point, let us recall the exterior Dirichlet-to-Neumann mapping for Helmholtz equation

$$\mathbb{T}_k : H^{1/2}(\partial B_R) \mapsto H^{-1/2}(\partial B_R) \quad \text{given by} \quad \mathbb{T}_k f = \frac{\partial \varphi}{\partial \nu} \quad \text{on } \partial B_R,$$

where  $\varphi \in H_{loc}^1(\mathbb{R}^d \setminus \overline{B_R})$  satisfies

$$\Delta \varphi + k^2 \varphi = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B_R} \quad \text{and} \quad \varphi = f \quad \text{on } \partial B_R$$

---

along with the radiation condition (11) where  $B_R = \{x \in \mathbb{R}^d : |x| < R\}$ . With the help of the Dirichlet-to-Neumann operator we can write (12) in its equivalent variational form: find  $u \in H^1(B_R)$  such that

$$\begin{aligned} - \int_{B_R} A \nabla u \cdot \nabla \bar{\varphi} - k^2 n u \bar{\varphi} dx + \int_{\partial B_R} \bar{\varphi} \mathbb{T}_k u ds \\ = - \int_D (I - A) \nabla v \cdot \nabla \bar{\varphi} - k^2 (1 - n) v \bar{\varphi} dx, \quad \forall \varphi \in H^1(B_R), \end{aligned} \quad (13)$$

where the boundary integral over  $\partial B_R$  is understood as the dual pairing between  $H^{1/2}(\partial B_R)$  and  $H^{-1/2}(\partial B_R)$ . It is clear by well-posedness that if we take  $v = \overline{\Phi(\cdot, y)}|_D$  then we have that the scattered field  $u^s(\cdot, y) = u$  for all  $x \in \mathbb{R}^d$ . Now define the source to trace operator

$$G : H^1(D) \mapsto L^2(C) \quad \text{given by} \quad Gv = u|_C$$

as well as the bounded linear operator

$$H : L^2(C) \mapsto H^1(D) \quad \text{given by} \quad (Hg)(x) = \int_C \overline{\Phi(\cdot, y)} g(y) ds_y|_D.$$

By superposition, we have that near field operator is given by  $N = GH$ . In order to use the theory developed in [18] and/or [20], we must construct a symmetric factorization of the near field operator. Therefore, we compute the adjoint of the operator  $H$  which is given in the following result.

**Theorem 3.1.** *The operator  $H^* : H^1(D) \mapsto L^2(C)$  is given by*

$$H^*v = \tilde{v}|_C \quad \text{for all } v \in H^1(D),$$

where  $\tilde{v} \in H_{loc}^1(\mathbb{R}^d)$  is the unique radiating solution to

$$- \int_{B_R} \nabla \tilde{v} \cdot \nabla \bar{\varphi} - k^2 \tilde{v} \bar{\varphi} dx + \int_{\partial B_R} \bar{\varphi} \mathbb{T}_k \tilde{v} ds = (v, \varphi)_{H^1(D)} \quad (14)$$

for all  $\varphi \in H^1(B_R)$ .

*Proof.* Let the radius of  $B_R$  be sufficiently large so that  $\bar{\Omega} \subset B_R$ . Given any  $v \in H^1(D)$  we can construct a unique radiating field  $\tilde{v} \in H_{loc}^1(\mathbb{R}^d)$  that satisfies (14). We now let  $w \in H_{loc}^1(\mathbb{R}^d)$  be defined as

$$w = \int_C \overline{\Phi(x, y)} g(y) ds_y \quad \text{for } x \in \mathbb{R}^d.$$



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Therefore, we have that

$$\begin{aligned}
(H^*v, g)_{L^2(C)} &= (v, Hg)_{H^1(D)} \\
&= - \int_{B_R} \nabla \tilde{v} \cdot \nabla \bar{w} - k^2 \tilde{v} \bar{w} dx + \int_{\partial B_R} \bar{w} \mathbb{T}_k \tilde{v} ds \\
&= - \int_{B_R \setminus \bar{\Omega}} \nabla \tilde{v} \cdot \nabla \bar{w} - k^2 \tilde{v} \bar{w} dx - \int_{\Omega} \nabla \tilde{v} \cdot \nabla \bar{w} - k^2 \tilde{v} \bar{w} dx + \int_{\partial B_R} \bar{w} \mathbb{T}_k \tilde{v} ds.
\end{aligned}$$

Using integration by parts on the volume integrals and the fact that  $w$  solves Helmholtz equation in  $\mathbb{R}^d \setminus C$  we have that

$$(H^*v, g)_{L^2(C)} = \int_C \tilde{v} \left( \frac{\partial}{\partial \nu} \bar{w}^{(-)} - \frac{\partial}{\partial \nu} \bar{w}^{(+)} \right) ds + \int_{\partial B_R} \bar{w} \frac{\partial}{\partial \nu} \tilde{v} - \tilde{v} \frac{\partial}{\partial \nu} \bar{w} ds.$$

Now using the jump relations for the normal derivative of the single layer potential for Helmholtz equation gives that

$$\frac{\partial}{\partial \nu} \bar{w}^{(-)} - \frac{\partial}{\partial \nu} \bar{w}^{(+)} = \bar{g} \quad \text{on } C.$$

Notice that both  $\tilde{v}$  and  $\bar{w}$  satisfy the radiation condition (11) therefore letting  $R \rightarrow \infty$  gives that

$$(H^*v, g)_{L^2(C)} = (\tilde{v}, g)_{L^2(C)},$$

proving the claim.  $\square$

Just as in [6] notice that (12), implies that

$$\Delta u + k^2 u = \nabla \cdot (I - A) \nabla (v + u) + k^2 (1 - n)(v + u) \quad \text{in } \mathbb{R}^d, \quad (15)$$

where  $u \in H_{loc}^1(\mathbb{R}^d)$  is the solution to (12) for a given  $v \in H^1(D)$ . The variational form of (15) is given by

$$\begin{aligned}
- \int_{B_R} \nabla u \cdot \nabla \bar{\varphi} - k^2 u \bar{\varphi} dx + \int_{\partial B_R} \bar{\varphi} \mathbb{T}_k u ds &= - \int_D (I - A) \nabla (v + u) \cdot \nabla \bar{\varphi} dx \\
&+ k^2 \int_D (1 - n)(v + u) \bar{\varphi} dx \quad \forall \varphi \in H^1(B_R).
\end{aligned}$$

Now appealing to the Riesz representation theorem, we can define the bounded linear operator  $T : H^1(D) \mapsto H^1(D)$  such that for all  $v \in H^1(D)$

$$(Tv, \varphi)_{H^1(D)} = - \int_D (I - A) \nabla (v + u) \cdot \nabla \bar{\varphi} - k^2 (1 - n)(v + u) \bar{\varphi} dx. \quad (16)$$

---

Notice for  $u \in H_{loc}^1(\mathbb{R}^d)$  the unique solution to (12) that by the definition of  $G$  we have that  $u|_C = Gv$ . Now by the definition of  $H^*$  and  $T$  we conclude that  $u|_C = H^*Tv$ . We conclude that  $Gv = H^*Tv$  for all  $v \in H^1(D)$ . Now recalling that  $N = GH$  gives the desired factorization.

**Lemma 3.1.** *The near field operator  $N : L^2(C) \mapsto L^2(C)$  for the scattering problem (10)-(11) has the factorization  $N = H^*TH$ .*

### 3.3 The solution to the inverse problem

The main goal of this section is to connect the support of  $D$  to the range of an operator defined by the measured near field operator. We make this connection by analyzing the factorization of the near field operator developed in the previous section. We recall from the previous section that we have the following factorization  $N = H^*TH$ . Under some appropriate assumptions on  $H$  and  $T$ , the factorization method gives that  $\text{Range}(H^*) = \text{Range}(N_{\sharp}^{1/2})$ , where we define

$$N_{\sharp} = |\Re(N)| + |\Im(N)| \quad \text{with} \quad \Re(N) = \frac{N + N^*}{2} \quad \text{and} \quad \Im(N) = \frac{N - N^*}{2i}.$$

Furthermore for a generic self-adjoint compact operator  $B$  on a Hilbert space  $U$ , by appealing to the Hilbert-Schmidt theorem we can define  $|B|$  in terms of the spectral decomposition such that

$$|B|x = \sum |\lambda_j|(x, \psi_j)\psi_j$$

for all  $x \in U$  where  $(\lambda_j, \psi_j) \in \mathbb{R} \times U$  is the orthonormal eigensystem of  $B$ .

#### 3.3.1 Analysis of the operator $H$

Now we turn our attention to showing that the operator  $H$  has the properties needed to apply Theorems A.1 and A.2. To this end, let us define the interior transmission problem as finding a pair  $(w, v) \in H^1(D) \times H^1(D)$  such that for a given pair  $(f, h) \in H^{1/2}(\partial D) \times H^{-1/2}(\partial D)$  satisfies

$$\nabla \cdot A \nabla w + k^2 n w = 0 \quad \text{and} \quad \Delta v + k^2 v = 0 \quad \text{in } D \quad (17)$$

$$w - v = f \quad \text{and} \quad \frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = h \quad \text{on } \partial D \quad (18)$$

**Definition 3.1.** The values of  $k \in \mathbb{C}$  for which the homogeneous interior transmission problem (i.e. (17)-(18) with  $(f, h) = (0, 0)$ ), has nontrivial solutions are called transmission eigenvalues for  $A$ ,  $n$  and  $D$ .

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It is known that under appropriate assumptions on the coefficients that the interior transmission problem (17)-(18) is Fredholm of index zero, i.e if  $k$  is not a transmission eigenvalue then (17)-(18) is well-posed (see for e.g. [1] and [5]). The following results are known for the transmission eigenvalue problem (see [1], [2] and [7])

1. If the scatterer is absorbing such that  $\Im(A) \leq 0$  and  $\Im(n) > 0$  in  $D$  then there are no real transmission eigenvalues.
2. If the scatterer is non-absorbing (i.e.  $\Im(A) = 0$  and  $\Im(n) = 0$ ) then the set of real transmission eigenvalues is at most discrete provided that  $A - I$  is positive (or negative) definite uniformly in  $D$ .
3. If the scatterer is non-absorbing then the set of real transmission eigenvalues is at most discrete provided that  $A - I$  is positive (or negative) definite uniformly and  $n - 1$  is positive (or negative) in a neighborhood of the boundary  $\partial D$ .

**Assumption 3.1.** *The wave number  $k \in \mathbb{R}$  is not a transmission eigenvalue for (17)-(18).*

We are now ready to connect the support of  $D$  to the range of  $H^*$ . Since we assume that the obstacle  $D$  is contained in the interior of  $\Omega$  we only need to consider sampling points  $z \in \Omega$ .

**Theorem 3.2.** *Under Assumption 3.1, we have that for  $z \in \Omega$*

$$\Phi(\cdot, z) \in \text{Range}(H^*) \quad \text{if and only if} \quad z \in D.$$

*Proof.* We start with the case where  $z \in \Omega \setminus \overline{D}$  and assume that  $v \in H^1(D)$  is such that  $H^*v = \Phi(\cdot, z)$ . By the definition of  $H^*$  there exists a  $\tilde{v} \in H_{loc}^1(\mathbb{R}^d)$  satisfying (14) which implies that

$$\Delta \tilde{v} + k^2 \tilde{v} = 0 \quad \text{in} \quad \mathbb{R}^d \setminus \overline{D} \quad \text{and} \quad \tilde{v} = \Phi(\cdot, z) \quad \text{on} \quad C$$

along with the Sommerfeld radiation condition (11). Now since  $z \in \Omega \setminus \overline{D}$ , we have that  $\Phi(\cdot, z)$  is in  $H_{loc}^1(\mathbb{R}^d \setminus \overline{\Omega})$ , and by the uniqueness of solutions to the exterior Dirichlet problem for Helmholtz equation, we have that  $\tilde{v} = \Phi(\cdot, z)$  in  $\mathbb{R}^d \setminus \overline{\Omega}$ . Therefore by Holmgren's theorem and unique continuation, we have that  $\tilde{v} = \Phi(\cdot, z)$  in  $\Omega \setminus (\overline{D} \cup \{z\})$ . We then conclude that  $\Phi(\cdot, z) \notin \text{Range}(H^*)$  since

$$\Phi(\cdot, z) \notin H^1(B_r(z)) \quad \text{and} \quad \tilde{v} \in H^1(B_r(z)),$$

for any disk  $B_r(z)$  centered at  $z$  of radius  $r > 0$ .

Now assume that  $z \in D$  then we have that  $\Phi(\cdot, z) \in H_{loc}^1(\mathbb{R}^d \setminus \overline{D})$ . Since  $k$  is not a transmission eigenvalue in  $D$  we have that there is a unique solution to the interior transmission problem (17)-(18) with

$$(f, h) = \left( \Phi(\cdot, z), \frac{\partial}{\partial \nu} \Phi(\cdot, z) \right) \quad \text{denoted} \quad (w_z, v_z) \in H^1(D) \times H^1(D).$$

---

Now define the function

$$u_z := \begin{cases} w_z - v_z & \text{in } D \\ \Phi(\cdot, z) & \text{in } \mathbb{R}^d \setminus \overline{D}. \end{cases}$$

Therefore we have that  $u_z \in H_{loc}^1(\mathbb{R}^d)$  with  $u_z|_C = \Phi(\cdot, z)|_C$  such that

$$\Delta u_z + k^2 u_z = \nabla \cdot (I - A) \nabla w_z + k^2(1 - n)w_z \text{ in } \mathbb{R}^d.$$

The variational form of the above equation gives that for all  $\varphi \in H^1(B_R)$  with  $\mathbb{T}_k$  being the Dirichlet to Neumann mapping

$$\begin{aligned} - \int_{B_R} \nabla u_z \cdot \nabla \overline{\varphi} - k^2 u_z \overline{\varphi} dx + \int_{\partial B_R} \overline{\varphi} \mathbb{T}_k u_z ds = \\ - \int_D (I - A) \nabla w_z \cdot \nabla \overline{\varphi} - k^2(1 - n)w_z \overline{\varphi} dx. \end{aligned} \quad (19)$$

Now let  $\phi_z \in H^1(D)$  be defined from the right hand side of (19) by appealing to the Riesz representation theorem, hence we have

$$- \int_{B_R} \nabla u_z \cdot \nabla \overline{\varphi} - k^2 u_z \overline{\varphi} dx + \int_{\partial B_R} \overline{\varphi} \mathbb{T}_k u_z ds = (\phi_z, \varphi)_{H^1(D)}.$$

Thus we now conclude that  $H^* \phi_z = \Phi(\cdot, z)$  by the definition of  $H^*$  giving the result.  $\square$

Next we show that  $H$  satisfies that necessary properties to apply Theorems A.1 and A.2.

**Theorem 3.3.** *The operator  $H : L^2(C) \mapsto H^1(D)$  is compact and injective.*

*Proof.* Notice that the mapping  $v \in H^1(D) \mapsto \tilde{v} \in H_{loc}^1(\mathbb{R}^d)$  is bounded and by the Trace theorem the mapping  $\tilde{v} \in H_{loc}^1(\mathbb{R}^d) \mapsto \tilde{v}|_C \in H^{1/2}(C)$  is also bounded. Now using that  $H^{1/2}(C)$  is compactly embedded in  $L^2(C)$  gives that  $H^*$  is compact and therefore so is  $H$ .

Now assume that  $g \in \text{Null}(H)$ , therefore we define

$$w = \int_C \Phi(x, y) \overline{g(y)} ds_y \quad \text{for } x \in \mathbb{R}^d.$$

We have that  $w$  is a radiating solution to Helmholtz equation in  $\mathbb{R}^d \setminus C$ , and by the definition of  $H$  we have that  $w = 0$  in  $D$ , and therefore by unique continuation we have that  $w = 0$  in  $\Omega$ . The continuity of the trace of  $w$  on  $C$  as well as the uniqueness of solutions to the exterior Dirichlet problem for Helmholtz equation implies that  $w = 0$  in  $B_R$  for any

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radius sufficiently large. Now assume that  $\bar{\Omega} \subset B_R$  using the jump relations for the normal derivative for the single layer potential for Helmholtz equation on  $C$  gives that

$$0 = \left( \frac{\partial}{\partial \nu} w^{(-)} - \frac{\partial}{\partial \nu} w^{(+)} \right) = \bar{g}$$

proving the claim.  $\square$

### 3.3.2 Analysis of the operator $T$

Next we analyze the properties of the middle operator  $T$  defined variationally by equation (16). The operator  $T$  is defined similarly to the operator used in the factorization of the far field operator discussed in [6], this gives that the analysis in this section follow from the methods employed in section 4 of [6]. Even though the analysis is similar we provide the details for sake of completeness. For later we recall that if  $u \in H_{loc}^1(\mathbb{R}^d)$  is a radiating solution to the exterior Helmholtz equation then we have that  $u$  satisfies that following expansion

$$u(x) = \frac{e^{ik|x|}}{|x|^{\frac{d-1}{2}}} \left\{ u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{|x|}\right) \right\} \quad \text{as } |x| \rightarrow \infty, \quad (20)$$

where  $\hat{x} := x/|x|$  and  $u^\infty(\hat{x})$  is the corresponding far field pattern (see [9] in  $\mathbb{R}^3$  and [2] in  $\mathbb{R}^2$ ). Now we consider the injectivity of the operator  $T$ .

**Theorem 3.4.** *The operator  $T : H^1(D) \mapsto H^1(D)$  is injective provided that one of the following conditions holds:*

1.  $\Im(A) = 0$ ,  $\Im(n) = 0$  in  $D$  and  $A - I$  is positive (or negative) definite uniformly in  $D$  and  $n - 1$  is negative (or positive) in  $D$ .
2.  $\Im(A) < 0$  in  $D$  and  $\int_D (n - 1) dx \neq 0$ .
3.  $\Im(A) \leq 0$  and  $\Im(n) > 0$  in  $D$ .

*Proof.* Assume that  $v \in \text{Null}(T)$ , therefore we have that the  $u \in H_{loc}^1(\mathbb{R}^d)$  defined as the unique solution of (12) satisfies for all  $\varphi \in H^1(B_R)$

$$-\int_{B_R} \nabla u \cdot \nabla \bar{\varphi} - k^2 u \bar{\varphi} dx + \int_{\partial B_R} \bar{\varphi} \mathbb{T}_k u ds = (Tv, \varphi)_{H^1(D)} = 0.$$

Which implies that  $u$  is an entire radiating solution to Helmholtz equation. We can then conclude that  $u = 0$  in  $\mathbb{R}^d$  and therefore by the definition of  $T$  we have that for all

---


$$\varphi \in H^1(D)$$

$$\int_D (A - I) \nabla v \cdot \nabla \overline{\varphi} - k^2(n - 1)v \overline{\varphi} \, dx = 0. \quad (21)$$

It is clear that if the first condition holds then the variational form given it (21) is coercive in  $H^1(D)$  which proves the claim by taking  $\varphi = v$ . Now if the second or third condition holds then letting  $\varphi = v$  and taking the imaginary part of (21) also proves infectivity.  $\square$

**Theorem 3.5.** *The operator  $\Im(T) : H^1(D) \mapsto H^1(D)$  satisfies the following:*

1.  $\Im(T)$  is non positive on  $H^1(D)$ .
2. If  $k$  satisfies assumption 3.1 then  $\Im(T)$  is negative on  $\overline{\text{Range}(H)}$ .
3. If  $\Im(A) = 0$  and  $\Im(n) = 0$  then  $\Im(T)$  is compact.

*Proof.* (i) By (12) we have that for any  $v_j \in H^1(D)$  there is a unique solution  $u_j \in H_{loc}^1(\mathbb{R}^d)$ . Now letting  $\phi_j = v_j + u_j$  and using (16) we have that

$$\begin{aligned} (Tv_1, v_2)_{H^1(D)} &= - \int_D (I - A) \nabla \phi_1 \cdot \nabla \overline{(\phi_2 - u_2)} - k^2(1 - n) \phi_1 \overline{(\phi_2 - u_2)} \, dx \\ &= - \int_D (I - A) \nabla \phi_1 \cdot \nabla \overline{\phi_2} - k^2(1 - n) \phi_1 \overline{\phi_2} \, dx \\ &\quad + \int_D (I - A) \nabla \phi_1 \cdot \nabla \overline{u_2} - k^2(1 - n) \phi_1 \overline{u_2} \, dx. \end{aligned}$$

Now by equation (12), we have that

$$\Delta u_1 + k^2 u_1 = \nabla \cdot (I - A) \nabla \phi_1 + k^2(1 - n) \phi_1 \quad \text{in } \mathbb{R}^d,$$

and therefore multiplying by  $\overline{u_2}$  and integrating by parts over  $B_R$  such that  $D \subset B_R$  gives that

$$\begin{aligned} - \int_{B_R} A \nabla u_1 \cdot \nabla \overline{u_2} - k^2 n u_1 \overline{u_2} \, dx + \int_{\partial B_R} \overline{u_2} \frac{\partial u_1}{\partial \nu} \, ds \\ = - \int_D (I - A) \nabla \phi_1 \cdot \nabla \overline{u_2} - k^2 \int_D (1 - n) \phi_1 \overline{u_2} \, dx. \end{aligned}$$

---

This gives that

$$\begin{aligned} (Tv_1, v_2)_{H^1(D)} = & - \int_D (I - A) \nabla \phi_1 \cdot \nabla \overline{\phi_2} - k^2(1 - n) \phi_1 \overline{\phi_2} dx \\ & + \int_{B_R} \nabla u_1 \cdot \nabla \overline{u_2} - k^2 u_1 \overline{u_2} dx - \int_{\partial B_R} \overline{u_2} \frac{\partial u_1}{\partial \nu} ds. \end{aligned} \quad (22)$$

Now take the imaginary part of equation (22) with  $u_2 = u_1$  and  $\phi_2 = \phi_1$ . Furthermore by letting  $R \rightarrow \infty$ , we obtain that

$$(\Im(T)v_1, v_1)_{H^1(D)} = \int_D \Im(A) \nabla \phi_1 \cdot \nabla \overline{\phi_1} - k^2 \Im(n) |\phi_1|^2 dx - k \int_{\mathbb{S}} |u_1^\infty|^2 ds(\hat{x}) \quad (23)$$

where the far field pattern  $u_1^\infty$  is defined by the asymptotic expansion in (20) and  $\mathbb{S} = \{x \in \mathbb{R}^d : |x| = 1\}$  is the unit circle or sphere. This proves that  $\Im(T)$  is non-positive since  $\Im(A) \leq 0$  and  $\Im(n) \geq 0$  in  $D$ .

(ii) We prove the claim by way of contradiction. To this end, let  $v \in \overline{\text{Range}(H)}$  and assume that  $(\Im(T)v, v)_{H^1(D)} = 0$ . We can then conclude that there is a sequence  $v_\ell \in \text{Range}(H)$  such that  $v_\ell \rightarrow v$  in  $H^1(D)$ , and let  $u_\ell \in H_{loc}^1(\mathbb{R}^d)$  be the sequence of the corresponding solutions of (12). The well-posedness of (12) gives that  $u_\ell$  is bounded in  $H_{loc}^1(\mathbb{R}^d)$ , therefore we can conclude that  $u_\ell$  converges to some  $u$  weakly in  $H_{loc}^1(\mathbb{R}^d)$ . Therefore, the limit  $u$  is a weak solution of the source problem (12). Now since

$$(\Im(T)v_\ell, v_\ell)_{H^1(D)} \longrightarrow 0 \quad \text{as } \ell \rightarrow \infty$$

equation (23) gives that  $u^\infty = 0$  whence Rellich's lemma and unique continuation implies that  $u = 0$  in  $\mathbb{R}^m \setminus \overline{D}$ . So we have that  $u^+ = 0$  and  $\frac{\partial u^+}{\partial \nu} = 0$  on  $\partial D$ . Therefore, the pair  $(u + v, v)$  are transmission eigenfunctions for  $D$ , but since  $k$  is not a transmission eigenvalue, we have that  $v = 0$ .

(iii) Now if we assume that  $\Im(A) = 0$  and  $\Im(n) = 0$  then

$$(\Im(T)v_1, v_2)_{H^1(D)} = -k \int_{\mathbb{S}} u_1^\infty(\hat{x}) \overline{u_2^\infty(\hat{x})} ds(\hat{x}),$$

now using the fact that the mapping  $v_j \mapsto u_j$  is bounded from  $H^1(D)$  to  $H_{loc}^1(\mathbb{R}^d)$  and  $u_j \mapsto u_j^\infty$  is compact from  $H_{loc}^1(\mathbb{R}^d)$  to  $L^2(\mathbb{S})$  (see for e.g. [18]) we can conclude that  $\Im(T)$  is compact.  $\square$

**Theorem 3.6.** *The operator  $\Re(T) : H^1(D) \mapsto H^1(D)$  satisfies the following*

- 
1. If  $\Re(A) - I$  is positive definite in  $D$  then  $\Re(T)$  is the sum of a compact operator and a self-adjoint coercive operator
  2. If  $(I - \Re(A) - \delta|\Im(A)|) > 0$  uniformly in  $D$  and  $(\Re(A) - \frac{1}{\delta}|\Im(A)|) \geq 0$  for some constant  $\delta > 0$  then  $-\Re(T)$  is the sum of a compact operator and a self-adjoint coercive operator

*Proof.* (i) Now assume that  $\Re(A) - I$  is positive definite in  $D$ . Therefore the variational form (22) gives that

$$\begin{aligned} (Tv_1, v_2)_{H^1(D)} &= \int_D (A - I) \nabla \phi_1 \cdot \nabla \overline{\phi_2} - k^2(n-1)\phi_1 \overline{\phi_2} dx \\ &\quad + \int_{B_R} \nabla u_1 \cdot \nabla \overline{u_2} - k^2 u_1 \overline{u_2} dx - \int_{\partial B_R} \overline{u_2} \mathbb{T}_k u_1 ds. \end{aligned} \quad (24)$$

where  $\mathbb{T}_k$  is the Dirichlet-to-Neumann mapping. Now define the bounded linear operators  $S$  and  $K : H^1(D) \mapsto H^1(D)$  via the Riesz representation theorem such that

$$\begin{aligned} (Sv_1, v_2)_{H^1(D)} &= \int_D (A - I) \nabla \phi_1 \cdot \nabla \overline{\phi_2} + \phi_1 \overline{\phi_2} dx \\ &\quad + \int_{B_R} \nabla u_1 \cdot \nabla \overline{u_2} + u_1 \overline{u_2} dx - \int_{\partial B_R} \overline{u_2} \mathbb{T}_k u_1 ds \\ (Kv_1, v_2)_{H^1(D)} &= -k^2 \int_D (n-1)\phi_1 \overline{\phi_2} dx - \int_D \phi_1 \overline{\phi_2} dx - \int_{B_R} (k^2 + 1)u_1 \overline{u_2} dx. \end{aligned}$$

It is clear that  $T = S + K$ . The compact embedding of  $H^1(D)$  into  $L^2(D)$  and  $H^1(B_R)$  into  $L^2(B_R)$  gives that the operator  $K$  is compact and therefore so is  $\Re(K)$ . We now show that  $\Re(S)$  is coercive on  $H^1(D)$ . To this end, taking the real part of the variational expression defining  $S$  gives that

$$\begin{aligned} (\Re(S)v_1, v_2)_{H^1(D)} &= \int_D (\Re(A) - I) \nabla \phi_1 \cdot \nabla \overline{\phi_2} + \phi_1 \overline{\phi_2} dx \\ &\quad + \int_{B_R} \nabla u_1 \cdot \nabla \overline{u_2} + u_1 \overline{u_2} dx - \int_{\partial B_R} \overline{u_2} \Re(\mathbb{T}_k) u_1 ds. \end{aligned}$$



---

To prove coercivity we recall that  $\phi_1 = v_1 + u_1$ , therefore

$$\begin{aligned} (\Re(S)v_1, v_1)_{H^1(D)} &= \int_D (\Re(A) - I) |\nabla(v_1 + u_1)|^2 + |(v_1 + u_1)|^2 dx \\ &\quad + \int_{B_R} |\nabla u_1|^2 + |u_1|^2 dx - \int_{\partial B_R} \overline{u_1} \Re(\mathbb{T}_k) u_1 ds. \end{aligned}$$

Using the fact that the real part of the Dirichlet-to-Neumann mapping  $\Re(\mathbb{T}_k)$  is non-positive (see [24] for details in  $\mathbb{R}^3$ ) we obtain that

$$(\Re(S)v_1, v_1)_{H^1(D)} \geq 0.$$

To complete the proof we proceed by way of contradiction, therefore consider a sequence  $v^n \in H^1(D)$  and the corresponding  $u^n \in H_{loc}^1(\mathbb{R}^d)$  such that

$$\|v^n\|_{H^1(D)} = 1 \quad \text{for which} \quad (\Re(S)v^n, v^n)_{H^1(D)} \rightarrow 0.$$

Therefore as  $n \rightarrow \infty$  we have that  $u^n \rightarrow 0$  in  $H^1(B_R)$  and  $v^n + u^n \rightarrow 0$  in  $H^1(D)$  since  $\Re(A) - I$  is positive definite. This implies that  $v^n \rightarrow 0$  in  $H^1(D)$  which contradicts the fact that  $\|v^n\|_{H^1(D)} = 1$  proving the claim.

(ii) We now consider the case where  $I - \Re(A)$  is positive definite but due to sign incompatibility of  $I - \Re(A)$  and the real part of the Dirichlet-to-Neumann mapping we must derive another expression to analysis  $-\Re(T)$ . To this end, using (16) and recall that  $\phi_j = v_j + u_j$ , we arrive at

$$\begin{aligned} (Tv_1, v_2)_{H^1(D)} &= - \int_D (I - A) \nabla \phi_1 \cdot \nabla \overline{v_2} - k^2(1 - n) \phi_1 \overline{v_2} dx \\ &= - \int_D (I - A) \nabla v_1 \cdot \nabla \overline{v_2} - k^2(1 - n) v_1 \overline{v_2} dx \\ &\quad - \int_D (I - A) \nabla u_1 \cdot \nabla \overline{v_2} - k^2(1 - n) u_1 \overline{v_2} dx. \end{aligned}$$

Now recall that for a given  $v_2 \in H^1(D)$  we have that  $u_2 \in H_{loc}^1(\mathbb{R}^d)$  satisfies

$$\nabla \cdot A \nabla u_2 + k^2 n u_2 = \nabla \cdot (I - A) \nabla v_2 + k^2(I - n) v_2 \quad \text{in } \mathbb{R}^d.$$

Therefore by multiplying the above differential equation by  $\overline{u_1}$  and applying integrating by

parts over  $B_R$  such that  $D \subset B_R$ , we obtain the following equation:

$$\begin{aligned} - \int_{B_R} A \nabla u_2 \cdot \nabla \overline{u_1} - k^2 n u_2 \overline{u_1} dx + \int_{\partial B_R} \overline{u_1} \frac{\partial u_2}{\partial \nu} ds \\ = - \int_D (I - A) \nabla v_2 \cdot \nabla \overline{u_1} - k^2 \int_D (1 - n) v_2 \overline{u_1} dx. \end{aligned} \quad (25)$$

We now take the conjugate of the above variational formula and replace the normal derivative on  $\partial B_R$  with the Dirichlet-to-Neumann mapping  $\mathbb{T}_k$  to obtain that

$$\begin{aligned} - (T v_1, v_2)_{H^1(D)} &= \int_D (I - A) \nabla v_1 \cdot \nabla \overline{v_2} - k^2 (1 - n) v_1 \overline{v_2} dx \\ &\quad + \int_{B_R} A \nabla u_1 \cdot \nabla \overline{u_2} - k^2 n u_1 \overline{u_2} dx - \int_{\partial B_R} u_1 \overline{\mathbb{T}_k u_2} ds \\ &\quad - \int_D (A - \overline{A}) \nabla u_1 \cdot \nabla \overline{v_2} - k^2 (n - \overline{n}) u_1 \overline{v_2} dx. \end{aligned}$$

In order to analyze  $\Re(T)$ , we must compute  $(T^* v_1, v_2)_{H^1(D)}$ . Then after some calculations, we obtain that

$$\begin{aligned} - (\Re(T) v_1, v_2)_{H^1(D)} &= \int_D (I - \Re(A)) \nabla v_1 \cdot \nabla \overline{v_2} - k^2 (1 - \Re(n)) v_1 \overline{v_2} dx \\ &\quad + \int_{B_R} \Re(A) \nabla u_1 \cdot \nabla \overline{u_2} - k^2 \Re(n) u_1 \overline{u_2} dx - \int_{\partial B_R} u_1 \overline{\Re(\mathbb{T}_k) u_2} ds \\ &\quad - i \int_D \Im(A) \nabla u_1 \cdot \nabla \overline{v_2} - k^2 \Im(n) u_1 \overline{v_2} dx + i \int_D \Im(A) \nabla v_1 \cdot \nabla \overline{u_2} - k^2 \Im(n) v_1 \overline{u_2} dx. \end{aligned}$$

(Notice that the complex constant  $i$  shows up in front of complex-valued terms.)

Now define the bounded linear operators  $S$  and  $K : H^1(D) \mapsto H^1(D)$  by the Riesz representation theorem such that

$$\begin{aligned} (S v_1, v_2)_{H^1(D)} &= \int_D (I - \Re(A)) \nabla v_1 \cdot \nabla \overline{v_2} + v_1 \overline{v_2} dx + \int_{B_R} \Re(A) \nabla u_1 \cdot \nabla \overline{u_2} dx \\ &\quad - \int_{\partial B_R} u_1 \overline{\Re(\mathbb{T}_k) u_2} ds - i \int_D \Im(A) \nabla u_1 \cdot \nabla \overline{v_2} dx + i \int_D \Im(A) \nabla v_1 \cdot \nabla \overline{u_2} dx \\ (K v_1, v_2)_{H^1(D)} &= (-\Re(T) v_1, v_2)_{H^1(D)} - (S v_1, v_2)_{H^1(D)}. \end{aligned}$$

---

Simple calculations show that in the definition of  $K$  there are only  $L^2$ -terms, hence  $K$  is a compact operator due to the compact embedding of  $H^1(D)$  into  $L^2(D)$  and  $H^1(B_R)$  into  $L^2(B_R)$ . Now, using that  $I - \Re(A) > 0$  and  $\Re(A) > 0$  along with the fact that the real part of the Dirichlet-to-Neumann mapping,  $\Re(\mathbb{T}_k)$  is non-positive, and by applying Young's inequality, we have that

$$\begin{aligned} (Sv_1, v_1)_{H^1(D)} &\geq \left( (I - \Re(A) - \delta|\Im(A)|) \nabla v_1, \nabla v_1 \right)_{L^2(D)} + (v_1, v_1)_{L^2(D)} \\ &\quad + \left( \left( \Re(A) - \frac{1}{\delta} |\Im(A)| \right) \nabla u_1, \nabla u_1 \right)_{L^2(D)} \geq C \|v_1\|_{H^1(D)}. \end{aligned}$$

Since there is a constant  $\delta > 0$  is such that

$$(I - \Re(A) - \delta|\Im(A)|) > 0 \text{ uniformly in } D \text{ and } \left( \Re(A) - \frac{1}{\delta} |\Im(A)| \right) \geq 0$$

we have proved the second part of the theorem.  $\square$

### 3.3.3 Characterization of the scattering object

We now have all we need to prove the main results of this paper. Using Theorems A.1 and A.2 along with Picard's criteria we will construct an indicator function to reconstruct the support of  $D$ . In this section we will consider two cases, i.e. for an absorbing and non-absorbing obstacle. We will state the main results separately for each case for simplicity.

**Theorem 3.7** (Range test for a non-absorbing media). *Assume that  $\Im(A) = 0$  and  $\Im(n) = 0$  in  $D$ . If  $A - I$  is positive (or negative) definite uniformly in  $D$  then for  $z \in \Omega$*

$$\Phi(\cdot, z) \in \text{Range}(N_{\sharp}^{1/2}) \quad \text{if and only if} \quad z \in D$$

where  $N_{\sharp} = |\Re(N)| + |\Im(N)|$ .

*Proof.* The result follows from the fact that Theorem A.1 along with Theorems 3.3, 3.5 and 3.6 gives that  $\text{Range}(N_{\sharp}^{1/2}) = \text{Range}(H^*)$  and then by appealing to Theorem 3.2 proves the claim.  $\square$

We now give a range test for an absorbing media.

**Theorem 3.8** (Range test for an absorbing media). *Assume that  $\Im(A) \leq 0$  and  $\Im(n) > 0$  in  $D$ . If either*

1.  $\Re(A) - I$  is positive definite in  $D$
2.  $(I - \Re(A) - \delta|\Im(A)|) > 0$  uniformly in  $D$  and  $(\Re(A) - \frac{1}{\delta} |\Im(A)|) \geq 0$  for some  $\delta > 0$

---

Then we have that for  $z \in \Omega$

$$\Phi(\cdot, z) \in \text{Range}(N_{\sharp}^{1/2}) \quad \text{if and only if} \quad z \in D$$

where  $N_{\sharp} = |\Re(N)| + |\Im(N)|$ .

*Proof.* The result follows from the fact that Theorem A.2 along with Theorems 3.3, 3.5 and 3.6 gives that  $\text{Range}(N_{\sharp}^{1/2}) = \text{Range}(H^*)$  and then by appealing to Theorem 3.3 proves the claim.  $\square$

Now let  $\lambda_i \in \mathbb{R}^+$  and  $\psi_i \in L^2(C)$  be an orthonormal eigensystem of the self-adjoint compact operator  $N_{\sharp}$  then applying Picard's criterion (see for e.g. Theorem 2.7 of [2]) to Theorems 3.7 or 3.8 gives the following.

**Corollary 3.1.** *Assume that the assumptions of Theorem 3.7 or 3.8 is satisfied then for  $z \in \Omega$*

$$W(z) = \left[ \sum_{i=1}^{\infty} \frac{|(\Phi(\cdot, z), \psi_i)|^2}{\lambda_i} \right]^{-1} > 0 \quad \text{if and only if} \quad z \in D.$$

Due to the rigorous range characterization given by the factorization method, we have the following uniqueness result for the inverse problem.

**Corollary 3.2** (Uniqueness for the Inverse Problem). *Assume that  $D_1$  and  $D_2$  are two penetrable anisotropic obstacles having material properties  $A_1, n_1$  and  $A_2, n_2$  that satisfy the assumptions for either Theorem 3.7 or 3.8 in the interior of  $D_1$  and  $D_2$ , respectively. If the corresponding scattered fields are such that  $u_1^s(x, y) = u_2^s(x, y)$  for all  $x, y \in C$  for a fixed wave number  $k$  (that satisfies assumption 3.1) then  $D_1 = D_2$ .*

**A numerical examples for unit ball in  $\mathbb{R}^2$ :** We will now give an explicit example of Theorem 3.7/3.8 where we assume that the coefficients are isotropic homogeneous in a disk of radius one. Therefore, the coefficient matrix is given by  $A = aI$  where both  $a$  and  $n$  are constant. Now notice that the scattering problem (10) can be written as find  $u \in H^1(B_1)$  and  $u^s \in H_{loc}^1(\mathbb{R}^2 \setminus \overline{B}_1)$  such that

$$\begin{aligned} a\Delta u + k^2 n u &= 0 \quad \text{in } B_1 \quad \text{and} \quad \Delta u^s + k^2 u^s = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{B}_1 \\ u^s - u &= -\overline{\Phi(x, y)} \quad \text{and} \quad \frac{\partial u^s}{\partial r} - a \frac{\partial u}{\partial r} = -\frac{\partial}{\partial r} \overline{\Phi(x, y)} \quad \text{on } \partial B_1, \end{aligned}$$

where  $u$  is the total field in  $B_1$  and  $u^s$  is the radiating scattered field in  $\mathbb{R}^2 \setminus \overline{B}_1$ . We make the ansatz that the solutions can be written as the following series

$$u^s(x, y) = \sum_{|m|=0}^{\infty} \alpha_m H_m^{(1)}(k|x|) e^{im\hat{x}} \quad \text{and} \quad u(x, y) = \sum_{|m|=0}^{\infty} \beta_m J_m \left( k \sqrt{\frac{n}{a}} |x| \right) e^{im\hat{x}}.$$

---

Here we adopt the notation that points in  $\mathbb{R}^2$  are given in by the polar representation

$$x = |x|(\cos(\hat{x}), \sin(\hat{x}))^\top,$$

where  $\hat{x} \in [0, 2\pi)$ . Recall that for  $|y| > |x|$  the fundamental solution has the expansion

$$\Phi(x, y) = \frac{i}{4} \sum_{|m|=0}^{\infty} H_m^{(1)}(k|y|) J_m(k|x|) e^{im(\hat{y}-\hat{x})}$$

(see for e.g. [9]) where  $J_m$  is the Bessel function and  $H_m^{(1)}$  is the first kind Hankel function of order  $m$ . We will assume that  $C = \partial B_2$  and therefore applying the boundary conditions we obtain that

$$\begin{bmatrix} H_m^{(1)}(k) & -J_m(k\sqrt{\frac{n}{a}}) \\ (H_m^{(1)}(k))' & -\sqrt{na}J'_m(k\sqrt{\frac{n}{a}}) \end{bmatrix} \begin{bmatrix} \alpha_m \\ \beta_m \end{bmatrix} = \frac{i}{4} H_m^{(2)}(2k) e^{-im\hat{y}} \begin{bmatrix} J_m(k) \\ J'_m(k) \end{bmatrix}.$$

Solving the linear system gives that

$$\alpha_m = \frac{i}{4} H_m^{(2)}(2k) e^{-im\hat{y}} \frac{J_m(k\sqrt{\frac{n}{a}}) J'_m(k) - \sqrt{na} J'_m(k\sqrt{\frac{n}{a}}) J_m(k)}{J_m(k\sqrt{\frac{n}{a}}) (H_m^{(1)}(k))' - \sqrt{na} J'_m(k\sqrt{\frac{n}{a}}) H_m^{(1)}(k)}$$

with  $H_m^{(2)}$  being the second kind Hankel function of order  $m$ . Simple manipulations give that the scattered field for  $x, y \in \partial B_2$  can be written as

$$u^s(x, y) = \frac{i}{4} \sum_{|m|=0}^{\infty} \sigma_m |H_m^{(1)}(2k)|^2 e^{im(\hat{x}-\hat{y})},$$

where the polar angles  $\hat{x}, \hat{y} \in [0, 2\pi)$  and we define

$$\sigma_m = \frac{J_m(k\sqrt{\frac{n}{a}}) J'_m(k) - \sqrt{na} J'_m(k\sqrt{\frac{n}{a}}) J_m(k)}{J_m(k\sqrt{\frac{n}{a}}) (H_m^{(1)}(k))' - \sqrt{na} J'_m(k\sqrt{\frac{n}{a}}) H_m^{(1)}(k)}.$$

Given the above expression for the scattered field we have that the near field operator can be written as  $N : L^2(0, 2\pi) \mapsto L^2(0, 2\pi)$

$$(Ng)(\hat{x}) = \frac{i}{4} \int_0^{2\pi} \sum_{|m|=0}^{\infty} \sigma_m |H_m^{(1)}(2k)|^2 e^{im(\hat{x}-\hat{y})} g(\hat{y}) d\hat{y}.$$

In our examples we approximate the near field operator by truncating the series. Let  $N_M$  be the truncated approximation of the near field operator for some  $M \in \mathbb{N}$  given by

$$(N_M g)(\hat{x}) = \frac{i}{4} \int_0^{2\pi} \sum_{|m|=0}^M \sigma_m |H_m^{(1)}(2k)|^2 e^{im(\hat{x}-\hat{y})} g(\hat{y}) d\hat{y}.$$

We apply Theorem 3.7/3.8 to the discretized version of the truncated near field operator  $\mathbf{N}_M$  where a simple 64 point Riemann sum approximation is used to discretize the integral. Therefore we will plot the function

$$W_M(z) = \left[ \sum_{i=1}^{64} \frac{|(\phi_z, \psi_i^M)_{\ell^2}|^2}{\lambda_i^M} \right]^{-1} \quad \text{for } \phi_z = \Phi(\cdot, z),$$

where  $\lambda_i^M$  and  $\psi_i^M$  are the eigenvalues and vectors of the matrix

$$(\mathbf{N}_M)_\# = |\Re(\mathbf{N}_M)| + |\Im(\mathbf{N}_M)|.$$

In the following examples we plot the contours of the function  $W_M(z)$  with  $M = 20$ .

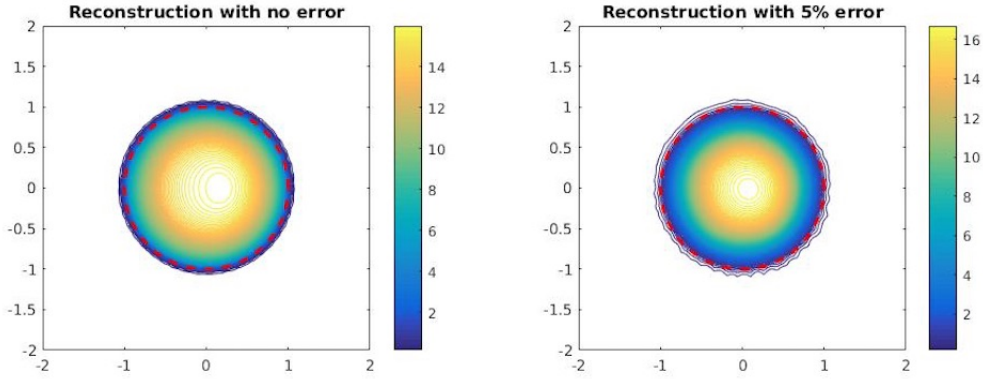


Figure 8: Reconstruction of the unit circle for  $k = 1$  via the factorization method where  $A = (1/2)I$  and  $n = 5$ . The dotted red line is the actual boundary of the unit circle.

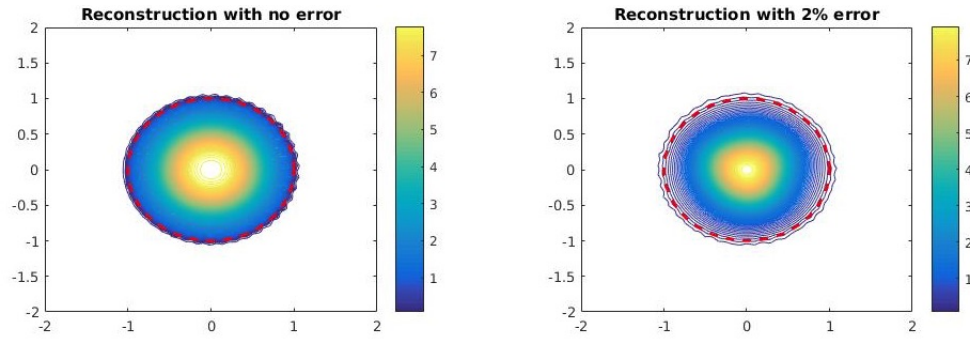


Figure 9: Reconstruction of the unit circle for  $k = 1$  via the factorization method where  $A = (3 - i)I$  and  $n = 1/4 + 2i$ . The dotted line is the actual boundary of the unit circle.

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## A Abstract theory of the factorization method

We now recall the main theoretical results from [20] and [18] which we use to develop the range test in the previous section. To this end, let  $X \subset U \subset X^*$  be a Gelfand triple with a Hilbert space  $U$  and a reflexive Banach space  $X$  such that the embedding is dense. Furthermore, let  $Y$  be a second Hilbert space and let  $N : Y \mapsto Y$ ,  $H : Y \mapsto X$  and  $T : X \mapsto X^*$  be linear bounded operators such that  $N = H^*TH$ .

**Theorem A.1.** (*Theorem 2.15 in [18]*) Assume that

1.  $H^*$  is compact with dense range.
2. There exists  $t \in [0, 2\pi]$  such that  $\Re(e^{it}T)$  is the sum of a compact operator and a self adjoint coercive operator.
3.  $\Im(T)$  is compact and non-negative (or non-positive) on the range of  $H$ .
4.  $\Re(e^{it}T)$  is injective or  $\Im(T)$  is strictly positive (or negative) on the closure of the range of  $H$ .

Then the operator  $N_{\sharp} = |\Re(e^{it}N)| + |\Im(N)|$  is positive, and the range of the operators  $H^* : X^* \mapsto Y$  and  $N_{\sharp}^{1/2} : Y \mapsto Y$  coincide.

**Theorem A.2.** (*Theorem 2.1 in [20]*) Assume that

1.  $H$  is compact and injective.
2.  $\Re(T)$  is the sum of a compact operator and a self adjoint coercive operator.
3.  $\Im(T)$  is non-negative (or non-positive) on  $X$ .

Moreover assume that either of the following is satisfied:

4.  $T$  is injective.
5.  $\Im(T)$  is strictly positive (or negative) on the null space of  $\Re(T)$ .

Then the operator  $N_{\sharp} = |\Re(e^{it}N)| + |\Im(N)|$  is positive, and the range of the operators  $H^* : X^* \mapsto Y$  and  $N_{\sharp}^{1/2} : Y \mapsto Y$  coincide.

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